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Modal Expansion Method for Eigensensitivity with Repeated Roots

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Extensions are made to the modal expansion eigensensitivity procedure for self-adjoint systems originally formulated by Fox and Kapoor. These allow the computation of sensitivity information for mode shapes associated with repeated roots and repeated root derivatives. The extensions follow a similar line to additions to Nelson's direct method proposed first by Ojalvo and later generalized independently by Mills-Curran and Dailey. The formulation based on modal expansion is simpler and more computationally efficient than the direct method. Although some accuracy is sacrificed by modal truncation, the method generally yields acceptable results for the primary application in optimization.

I. Introduction

EFFICIENT and accurate computation of modal parameter sensitivity to structural model changes is necessary for diverse problems such as optimization and system identification. For example, many current spacecraft designs involve constraints on dynamic deflections associated with mission critical electro-optical pointing requirements. Structural optimization is a tool which can help meet these constraints and minimize structural weight.[1] Furthermore, matching finite element model predictions to measured test data is essential for verifying launch load capabilities and developing accurate plant models for active vibration control systems.[2] Optimization and physical parameter identification schemes are also becoming important in the design and analysis of structures such as disk drives, high-performance aircraft, and even heavy earth-moving equipment.

Modal parameter sensitivity analysis is especially significant because modal methods are widely employed for dynamic analysis. In addition, the common use of modal methods in dynamic testing means that system identification techniques must usually have as a goal the matching of modal parameters. The modal parameters of interest are frequency and mode shape. Modal damping can also be determined from tests and predicted approximately by finite element models using strain energy techniques. However, in the strictest mathematical sense damping forces necessitate the use of complex modes rather than the real normal modes which are the subject of this paper.

Based on the assumption that the lowest natural frequencies of a structure are indicative of overall flexibility, the design of structures can frequently be undertaken by optimizing weight subject to minimum frequency constraints.[3] This allows one to employ eigenvalue sensitivity analysis alone, which is straightforward and increasingly available in commercial finite element packages.[4] The number of full finite element analyses can be reduced by using accurate first-order approximations such as convoluted Taylor series,[3] which facilitates a convex approximation to the original constraint or objective function.

Response minimization and parameter identification techniques require both eigenvalue and eigenvector sensitivity information. Methods for computing eigenvector sensitivity can be either direct or based on modal expansions, both of which are discussed by Fox and Kapoor.[5] The modal expansion method is quite simple and generally more efficient than direct methods when modal truncation is employed. A method to improve the accuracy of the modal expansion technique has been derived by Wang.[6] In addition to the modal basis, it employs a static correction term, similar to the mode acceleration method used to improve accuracy of structural dynamic simulations. The accuracy using Wang's explicit method has been found to approach the exact solution with a small number of retained modes.[7] Wang's results using the implicit method indicate even better performance.

Direct methods may be as efficient when the sensitivity of only a few modes is of interest, however, in practice a large number of modes is usually required for dynamic analysis. The direct method presented in Ref. 5 requires, for an n-degree-of-freedom system, the multiplication of an $n + 1$ by n matrix by its transpose for each eigenvector sensitivity calculation. A resultant $n \times n$ system of equations with no bandedness must then be solved.

A more efficient direct method was introduced by Nelson.[8] This method preserves bandedness and does not require a large matrix multiplication. However, the method still requires solution of an nth-order system for each eigenvector for which sensitivity information is desired. A direct method based on the Moore-Penrose generalized inverse was formulated by Chen and Wei.[9] An eigensensitivity approach employing singular value decomposition (SVD) is discussed by Lim et al.[10] Of the exact approaches (those not employing modal expansions) Nelson's method[8] with extensions to the case of repeated roots has received the majority of recent research attention since it appears to be the most computationally efficient.

Unfortunately, both Fox and Kapoor's modal expansion method[5] and Nelson's direct method[8] produce singular equations when solving for eigenvectors corresponding to repeated eigenvalues. Repeated roots occur often in structures having redundant or symmetrical features. Chen and Pan[11] introduced methodology which allows sensitivity computations for repeated eigenvalues. They made an initial attempt at eigenvector sensitivity using a modal expansion approach, with apparently incomplete results. A more complete treatment of the repeated roots case using a modal expansion has been formulated by Juang et al.[12] That work treated the case of a general, nondefective matrix. The solution was in terms of a coupled set of matrix equations (singular) in the modal admixture coefficients for the eigenvector derivative. Additional constraint equations were appended to allow solution of the singular equations. The algorithm presented in this paper includes a specific formula for all of the admixture coefficients, without the need to solve a set of matrix equations. The SVD approach cited earlier has an inherent capability for treating the case of repeated roots, but does not address repeated root derivatives. Attention has been devoted recently to extensions to Nelson's method to allow treatment of the repeated roots case. Ojalvo[13,14] produced an extension to Nelson's method which applies in some special cases, although a simplifying assumption was made which is not generally true. Both Mills-Curran[15] and Dailey[16] independently derived corrected

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versions of extensions to Nelson's method which apply in cases when eigenvalues are repeated but eigenvalue derivatives are distinct. Mills-Curran's algorithm[17] included a procedure for avoiding singularities in choosing a partition of the eigenproblem used in solving for the eigenvector derivative, which was not included in Dailey's algorithm. Dailey also presented some results for the case of repeated eigenvalue derivatives.[16]

Most design optimizations in practice are performed on finite element models with hundreds, if not thousands, of degrees of freedom. Considering that solutions to these problems usually involve modal truncation, the use of a modal expansion technique to determine eigenvector sensitivities is consistent. The objective herein centers on extending Fox and Kapoor's modal expansion method to allow computation of eigenvector derivatives for the case in which eigenvalues are repeated with distinct eigenvalue derivatives and also the case where eigenvalues and eigenvalue derivatives are both repeated. The derivations follow similar lines to those in Ref. 12.

II. Definition of the Eigenvalue Problem

The real, symmetric structural eigenproblem is defined by

$$K\Phi = M\Phi\Lambda \tag{1}$$

where the stiffness and mass matrices K and M are real, symmetric positive definite or positive semidefinite matrices of order n. It is well known that the eigenvalues and eigenvectors are real and that the eigenvalues are non-negative. Λ is a diagonal matrix of n eigenvalues λ_i, and Φ is the modal matrix whose columns are the eigenvectors ϕ_i. It is assumed that the eigenvalues are arranged in ascending order, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Given that the mass matrix is positive definite, the eigenvectors may be orthonormalized with respect to the mass matrix such that

$$\Phi^T M \Phi = I \tag{2}$$

where I is an identity matrix of order n. Consequently, the eigenvectors are orthogonal with respect to the stiffness matrix as follows:

$$\Phi^T K \Phi = \Lambda \tag{3}$$

The preceding relations are well known and are shown for the case of nonrepeated roots in Sec. 13.1 of Ref. 18. For the case of repeated roots, use is typically made of theorems relating to the eigenproblem in standard form,

$$D\Psi = \Psi\Lambda \tag{4}$$

Since the mass matrix is positive definite, it can be decomposed into lower triangular form,

$$M = LL^T \tag{5}$$

The transformation

$$\Psi = L^T\Phi \tag{6}$$

allows the structural eigenproblem to be cast in standard form with D symmetric. It is shown, for example, in Theorem 9.3 of Ref. 19 that symmetry of D allows the eigenvectors Ψ to be chosen to form an orthonormal set even for the case of repeated eigenvalues. Back transforming, it is simple to show that the orthogonality relations of Eqs. (2) and (3) will also hold.

The orientation of a set of eigenvectors corresponding to a repeated eigenvalue is arbitrary within their subspace. Given an eigenvalue λ_i of multiplicity r_i, we may define a modal matrix Φ_i whose columns are made up of these eigenvectors, i.e.,

$$\Phi_i \equiv [\phi_i, \phi_{i+1}, \ldots, \phi_{i+r_i-1}] \tag{7}$$

Φ_i is assumed to have been arranged so as to satisfy the orthogonality relations (2) and (3). The eigenvectors corresponding to λ_i span an invariant subspace of dimension r_i within $\Re^n$. Any linear combination of these eigenvectors is also an eigenvector corresponding to λ_i. Transformation of eigenvectors within the subspace is given by

$$\bar{\Phi}_i = \Phi_i A_i \tag{8}$$

Conditions on A_i to maintain orthogonality are obtained by enforcing orthogonality on the reoriented eigenvectors $\bar{\Phi}_i$,

$$\bar{\Phi}_i^T M \bar{\Phi}_i = A_i^T \Phi_i^T M \Phi_i A_i = A_i^T A_i \tag{9}$$

$$\bar{\Phi}_i^T K \bar{\Phi}_i = A_i^T \Phi_i^T K \Phi_i A_i = \lambda_i A_i^T A_i \tag{10}$$

Continued satisfaction of orthogonality requires that the transformation matrix A_i must itself be orthonormal, i.e., that $A_i^T A_i = I$, where I is an identity matrix of order r_i. Orthonormal transformations preserve both lengths of vectors and inner products between vectors, as shown in Theorem 8.8 of Ref. 19. Interpreted geometrically, this means that rotations and reflections may be applied to the eigenvectors within the invariant subspace spanned by Φ_i.

A result concerning small motions within the invariant subspace is demonstrated which will be useful in interpreting sensitivity. Consider Eq. (8) where the transformation is a perturbation $A_i = I + \delta A_i$, the elements of δA_i being small compared to 1. The orthonormality condition on A_i requires that

$$A_i^T A_i = (I + \delta A_i)^T (I + \delta A_i) = I + \delta A_i^T + \delta A_i + \delta A_i^T \delta A_i = I \tag{11}$$

For infinitesimal perturbations, the quadratic term may be ignored leading to

$$\delta A_i^T + \delta A_i = 0 \tag{12}$$

One sees that skew-symmetric perturbations, ($\delta A_i^T = -\delta A_i$) which may be interpreted geometrically as small rigid body rotations, automatically satisfy the orthonormality condition. However, symmetric perturbations ($\delta A_i^T = \delta A_i$), which represent deformation of the subspace, will destroy orthogonality. (A full reflection will preserve orthogonality of the eigenvectors, despite it being a symmetric transformation. Note, however, that small reflections do not exist. In a differential sense, a reflection does require deformation of the space, as vector components shrink to zero in the direction of the plane of reflection and then expand in the opposite direction.)

III. Eigenvalue Derivatives and Unique Eigenvectors

As shown in Refs. 11 and 13, uniqueness of eigenvectors in the case of repeated roots can sometimes be established based on sensitivity information. Although an eigenvalue may be repeated, the eigenvalue derivatives with respect to a parameter may not all be the same. One can then develop unique eigenvectors associated with each of the unique eigenvalue derivatives. Among the eigenvalue derivatives which remain repeated, unique eigenvectors may not be ascertained on this basis. Further considerations such as second derivative of the eigenvalue may be employed to define unique eigenvectors as described in Ref. 12, but this may not be meaningful for practical applications.

A method to determine the eigenvalue derivatives and unique eigenvectors for repeated eigenvalues is shown in Ref. 15. Consider the defining eigenproblem, Eq. (1), with the invariant subspace $\bar{\Phi}_i$ employed instead of the entire eigenspace. The matrix Λ_i will consist of the single eigenvalue λ_i on each of its diagonal terms. The mass and stiffness matrices are assumed to be explicit functions of some design parameter D. The eigenvalues and vectors are implicit functions of this parameter by nature of the constraint Eq. (1). Taking the derivative of Eq. (1) with respect to the design parameter, premultiplying by the transpose of Φ_i and rearranging terms, the result is

$$\Phi_i^T(K' - \lambda_i M')\bar{\Phi}_i - \Phi_i^T M \bar{\Phi}_i \Lambda_i' + \Phi_i^T (K - \lambda_i M)\bar{\Phi}_i' = 0 \tag{13}$$

where derivatives are denoted with a prime. The last term in Eq. (13) is zero due to symmetry of K and M and satisfaction of the defining eigenproblem (1). Substituting Eq. (8) and taking advantage of mass orthonormality one obtains

$$[\Phi_i^T(K' - \lambda_i M')\,\Phi_i]\,A_i = A_i\Lambda_i' \tag{14}$$

This eigenvalue equation will define a set of unique eigenvalue derivatives Λ_i' if they exist, i.e., if the eigenvalues separate on differentiation. Corresponding to these will be a set of unique eigenvectors corresponding to the bifurcated eigenvalues, obtained by substituting A_i in Eq. (8). If repeated eigenvalue derivatives exist, then within that invariant subspace the eigenvectors will remain indeterminate with respect to rotations and reflections. Note that if eigenvalues are unique to begin with, then Eq. (14) is a scalar equation and will produce the eigenvalue derivative λ_i' directly.

IV. Modal Expansion Method for Eigenvector Derivatives

A method of approximating the eigenvector derivative as a linear sum of the eigenvectors was first derived by Fox and Kapoor.[5] The assumption is

$$\phi_i' \approx \sum_{j=1}^{m} \phi_j c_{ji} \tag{15}$$

where m can vary up to the problem order n, in which case the approximation becomes exact. The basic eigenvalue problem (1) can be written more compactly as

$$(K - \lambda_i M)\phi_i \equiv F_i\phi_i = 0 \tag{16}$$

Differentiating Eq. (16) produces a defining equation for the eigenvector derivatives.

$$F_i'\phi_i + F_i\phi_i' = 0 \tag{17}$$

Using the foregoing relations, the admixture coefficients c_{ji} may now be determined on a case by case basis for each of the eigenvalue multiplicity conditions.

Case 1: Unique Eigenvalues ($\lambda_i \neq \lambda_j$)

When the multiplicity of the eigenvalue is one, the modal approximation (15) may be substituted in Eq. (17) and premultiplied by ϕ_j^T. As shown in Ref. 5, application of orthogonality conditions allows one to solve for the off-diagonal admixture coefficients

$$c_{ji} = \frac{\phi_j^T(K' - \lambda_i M')\phi_i}{(\lambda_i - \lambda_j)}; \qquad (\lambda_i \neq \lambda_j) \tag{18}$$

As long as the eigenvalues remain unique, Eq. (18) can be solved. Note that the formula provides for a degree of convergence related to the eigenvalue separation. For instance, if one desires the derivative of a low-frequency mode i, the contributions of higher frequency modes j become relatively less important.

Case 2: Same Mode ($i = j$)

Differentiation of the mass normalization condition (2) on mode i produces a defining equation for the diagonal admixture coefficients. After substitution of the modal approximation (15) and application of orthonormality conditions one obtains

$$c_{ii} = \frac{-\phi_i^T M'\phi_i}{2} \tag{19}$$

This formula simply reflects the fact that one must scale the mode shape as one varies the mass to preserve the normalization factor.

Case 3: Unique Eigenvalue Derivatives ($\lambda_i = \lambda_j$, $\lambda_i' \neq \lambda_i'$)

Solution for the admixture coefficients corresponding to repeated eigenvalues but nonrepeated eigenvalue derivatives will follow along the lines of the extensions to Nelson's method proposed first by Ojalvo[13] and later corrected by Mills-Curran[15] and Dailey.[16] The use of the modal expansion approximation results in a simpler, easier to follow derivation. A defining equation is produced through the differentiation of Eq. (17)

$$F_i''\phi_i + 2F_i'\phi_i' + F_i\phi_i'' = 0 \tag{20}$$

where ϕ_i is taken to be among a subset of eigenvectors with repeated eigenvalues and distinct eigenvalue derivatives. It is assumed that the eigenvectors have been reoriented according to Eqs. (8) and (14). Bars over the symbols have been omitted for clarity. Premultiplication by ϕ_j^T gives

$$\phi_j^T F_i''\phi_i + 2\phi_j^T F_i'\phi_i' + \phi_j^T F_i\phi_i'' = 0 \tag{21}$$

Note that if $\lambda_j = \lambda_i$, then $F_i = F_j$. Hence, $\phi_j^T F_i = 0$ due to symmetry of F_i and satisfaction of Eq. (16). The third term on the left side of Eq. (21) thus disappears. Substitution of the modal expansion approximation gives

$$\underbrace{\phi_j^T F_i''\phi_i}_{T_{ji}^1} + \underbrace{2\phi_j^T F_i'\sum_k \phi_k c_{ki}}_{T_{ji}^2} = 0 \tag{22}$$

The two terms of Eq. (22) are now examined individually.

Term T_{ji}^1: Expanding F_i'' gives

$$T_{ji}^1 = \phi_j^T[K'' - 2\lambda_i'M' - \lambda_i M'']\phi_i - \lambda_i''\phi_j^T M\phi_i \tag{23}$$

As a result of orthonormality, we can replace the last term with the Kronecker delta, giving

$$T_{ji}^1 = \phi_j^T[K'' - 2\lambda_i'M' - \lambda_i M'']\phi_i - \lambda_i''\delta_{ij} \tag{24}$$

Term T_{ji}^2: Expanding F_i' gives

$$T_{ji}^2 = 2\phi_j^T[K' - \lambda_i M' - \lambda_i' M]\sum_k \phi_k c_{ki} \tag{25}$$

Adding and subtracting $\lambda_j' M$

$$T_{ji}^2 = 2\phi_j^T[K' - \lambda_i M' - \lambda_j' M + (\lambda_j' - \lambda_i')M]\sum_k \phi_k c_{ki} \tag{26}$$

Use of the equality $\lambda_i = \lambda_j$ and application of mass orthonormality to the last term of Eq. (26) allows simplification to the form

$$T_{ji}^2 = 2\phi_j^T[K' - \lambda_j M' - \lambda_j' M]\sum_{k=1}^{m} \phi_k c_{ki} + 2(\lambda_j' - \lambda_i')c_{ji} \tag{27}$$

Noting that the term in brackets in Eq. (27) equals F_j', terms for which $\lambda_k = \lambda_j$ can be shown to drop out of the summation. If Eq. (17) is premultiplied by ϕ_k^T, the result is

$$\phi_k^T F_j'\phi_j = -\phi_k^T F_j\phi_j' \tag{28}$$

If $\lambda_k = \lambda_j$, then $F_k = F_j$. As a result of satisfaction of the eigenvalue problem ($F_k\phi_k = 0$) and symmetry of F, we obtain $\phi_k^T F_j = 0$. Since the right side of Eq. (28) equals zero, $\phi_k^T F_j'\phi_j = 0$ as does its transpose $\phi_j^T F_j'\phi_k$. Therefore, all components of the summation in Eq. (27) for which $\lambda_k = \lambda_j$ are zero. Noting that $\lambda_j = \lambda_i$ and expanding, one obtains

$$T_{ji}^2 = 2\phi_j^T[K' - \lambda_i M']\sum_{\substack{k=1\\ \lambda_k \neq \lambda_i}}^{m} \phi_k c_{ki} - 2\lambda_j'\phi_j^T M\sum_{\substack{k=1\\ \lambda_k \neq \lambda_i}}^{m} \phi_k c_{ki}$$
$$+ 2(\lambda_j' - \lambda_i')c_{ji} \tag{29}$$

Because of mass orthonormality, the second term in Eq. (29) equals zero when $\lambda_k \neq \lambda_i$, so it drops out. Equation (22) finally simplifies to

$$\phi_j^T [K'' - 2\lambda_i' M' - \lambda_i M'']\phi_i + 2\phi_j^T [K' - \lambda_i M'] \sum_{\substack{k=1 \\ \lambda_k \neq \lambda_i}}^{m} \phi_k c_{ki} - \lambda_i'' \delta_{ij} + 2(\lambda_j' - \lambda_i') c_{ji} = 0 \tag{30}$$

The c_{ki} terms can be determined from Eq. (18) using k in place of j. Equation (30) may thus be used to solve for the admixture coefficients c_{ji} in Case 3, $\lambda_i = \lambda_j$, $\lambda_i' \neq \lambda_j'$, yielding

$$c_{ji} = \frac{\phi_j^T [K'' - 2\lambda_i' M' - \lambda_i M'']\phi_i + 2\phi_j^T [K' - \lambda_i M'] \sum_{k=1, \lambda_k \neq \lambda_i}^{m} \phi_k c_{ki}}{2(\lambda_i' - \lambda_j')} \tag{31}$$

[The authors are indebted to William Mills-Curran for pointing out a flaw in our original formulation for this case. This led to a missing factor of 2 in the numerator of Eq. (31). The flawed result was presented in Eq. (5) of Ref. 20. The current formulation incorporates his suggestion.]

Having used Eq. (30) to solve for the desired eigenvector derivative admixture coefficients in case 3, we may also use it in the case where $i = j$, to solve for the second eigenvalue derivative, obtaining

$$\lambda_i'' = \phi_i^T [K'' - 2\lambda_i' M' - \lambda_i M'']\phi_i + 2\phi_i^T [K' - \lambda_i M'] \sum_{\substack{k=1 \\ \lambda_k \neq \lambda_i}}^{m} \phi_k c_{ki} \tag{32}$$

Inserting the known value for c_{ki}, this may be written

$$\lambda_i'' = \phi_i^T [K'' - 2\lambda_i' M' - \lambda_i M'']\phi_i + 2 \sum_{\substack{k=1 \\ \lambda_k \neq \lambda_i}}^{m} [\phi_i^T (K' - \lambda_i M')\phi_k]^2 / (\lambda_i - \lambda_k) \tag{33}$$

This result may be checked against other derivations for second eigenvalue derivative. It is interesting to note the existence of a term which is quadratic in the derivative of the stiffness and mass matrices.

Case 4: Nonunique Eigenvalue Derivatives ($\lambda_i = \lambda_j$, $\lambda_i' = \lambda_j'$)

When the derivatives of repeated eigenvalues are also repeated, other considerations may be employed to determine the off-diagonal admixture coefficients. Rather than taking third or higher derivatives of the eigenvalue problem to produce a defining equation, consider the constraint equation

$$\phi_j^T M \phi_i = \delta_{ij} \tag{34}$$

Taking the derivative of this gives

$$(\phi_j')^T M \phi_i + \phi_j^T M' \phi_i + \phi_j^T M \phi_i' = 0 \tag{35}$$

Substitution of the modal expansion then gives

$$(\phi_k c_{kj})^T M \phi_i + \phi_j^T M' \phi_i + \phi_j^T M \phi_k c_{ki} = 0 \tag{36}$$

where a summation is implicitly performed over values of k from 1 to m. Application of mass orthonormality allows this to be expressed in terms of the Kronecker delta as

$$c_{kj} \delta_{ki} + \phi_j^T M' \phi_i + \delta_{jk} c_{ki} = 0 \tag{37}$$

which then reduces to

$$c_{ij} + c_{ji} = -\phi_j^T M' \phi_i \tag{38}$$

This equation gives information only about the symmetric portion of C. Since the antisymmetric portion has been shown to represent arbitrary infinitesimal rotations, we see that these remain arbitrary and may thus be ignored. Assuming symmetry for C allows both diagonal and off-diagonal terms to be obtained

$$c_{ij} = \frac{-\phi_j^T M' \phi_i}{2} \tag{39}$$

It should be noted that this is the same equation as used for the diagonal c_{ii} terms by Fox and Kapoor. In this case it may be extended to provide information regarding eigenvector derivative components which would be unavailable without taking third or higher derivatives of the eigenvalue equation.

Improvement of Accuracy

To account for the truncated modes Wang[6] has derived both an explicit and implicit method for calculating a static correction using the residual flexibility of the truncated modes. In the explicit method the corrected eigenvector derivative takes the form

$$\phi_i' \approx \sum_{j=1}^{m} \phi_j c_{ji} + \left(K^{-1} - \sum_{j=1}^{m} \frac{\phi_j \phi_j^T}{\lambda_j} F_i' \phi_i \right) \tag{40}$$

The implicit method provides a different coefficient on the correction term, giving better accuracy. In either case, the correction does not affect the computation of the admixture coefficients for the included modes.

V. Numerical Example

Model Description

The truncated quadrapod shown in Fig. 1 is employed as an example. This structure is dynamically rich, having eight modes between 19 and 32 Hz, with only three unique eigenvalues. Of these modes, 1 and 2 were considered repeated, as were modes 4–8. Proper selection of design variables can provide repeated modes with repeated and/or distinct eigenvalue derivatives. The model has 30 active degrees of freedom, and 18 out of the 30 modes were included in the modal approximations.

The quadrapod legs are 274.3-cm-long square tubes whose initial properties are described in Table 1. Each is modeled in MSC/NASTRAN (by MacNeal-Schwendler Corporation) with two equal length bar elements. The legs are fixed at ground on a 58.4-cm-radius circle, connecting at the top to a rigid plate having a 7.62-cm radius. Each leg has a 0.453-kg mass located midway along its length. There is a 0.906-kg mass with a 10.16-cm offset above the top plate.

A simple optical performance metric was built into the model as shown in Fig. 1. Two line-of-sight errors, LOS_x and LOS_y, were defined as the x and y motions on the floor of a light pencil emanating from the top plate. Also monitored were the lateral translations, δ_x and δ_y, of the top plate.

Two design variables D were initially investigated. For computation they were normalized by their initial values, giving $\bar{D}_i \equiv D_i/D_i^0$. D_1 is the tube wall thickness along the entire length of just one of the legs. Numerous repeated frequencies result when this thickness equals that of the other three legs ($\bar{D}_1 = 1$). However, some of the eigenvalue derivatives are distinct because changes to D_1 cause unsymmetric changes to the structure. D_1 is capable of exercising cases 3 and 4 for a single repeated eigenvalue. D_2 was taken to be the tube wall thickness of the upper elements on all four legs. An initial value of $\bar{D}_2 = 1$ also gives numerous repeated frequencies. However, the modes stay repeated as D_2 changes since symmetry remains unbroken. D_2 will not exercise case 3 and, therefore, was not investigated further. The first 18 eigenvalues for the initial design and their derivatives with respect to both design variables are shown in Table 2.

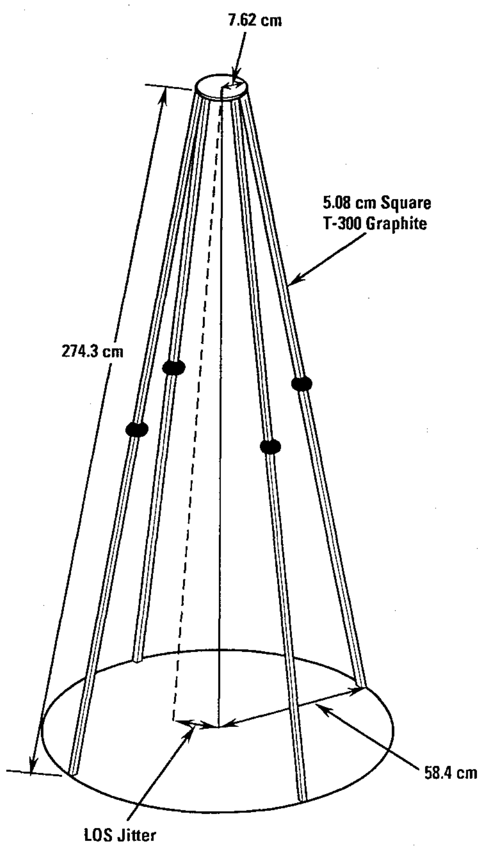

Fig. 1 Quadrapod geometry.

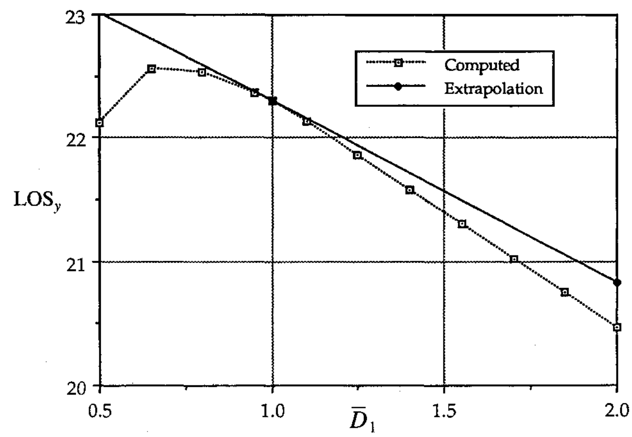

Fig. 2 LOS$_y$ in mode 1 as a function of $\bar{D}_1$.

Eigenvector Derivatives

Eigenvector derivatives with respect to the first design variable in the four monitored displacement components are shown in Table 3. The simple modal expansion without a static correction term was employed. Modes 1–8 with the exception of mode 3 are shown, since these represent the first two sets of repeated eigenvalues. Mode 3 involves torsion of the quadrapod and does not couple with the lateral bending modes, which give rise to the repeatedness of this example. Eigenvector derivative components for modes 1 and 2 possess modal contributions satisfying cases 1–3. Modes 1 and 2 have repeated eigenvalues and their eigenvalue derivatives are not repeated (case 3); other modes are used in the summation which have different eigenvalues (case 1) and, of course, the same mode (case 2) also pertains. Modes 4–6 exercise all four cases, since there is the existence of repeated roots and also repeated derivatives among modes 4–6. Modes 7 and 8 exercise cases 1–3 for similar reasons to modes 1 and 2. The first 18 modes were used in the eigenvector derivative computations. The ratio of the eigenvalue for modes 1 and 2 to the eigenvalue of the highest mode retained in the summations is 0.0015, and for modes 4–8 the ratio is 0.0037. This ratio is indicative of the convergence in the estimate to be achieved using Eq. (18).

To evaluate the effectiveness of the modal approximation, Table 3 also contains a comparison to eigenvector derivatives obtained by finite difference computation. The finite difference vector was calculated by subtracting the reoriented eigenvectors at $\bar{D}_1 = 1$ from those at $\bar{D}_1 = 1.02$ and dividing by 0.02. The comparison between the modal expansion sensitivity and the finite difference is reasonably good, especially if one compares any discrepancy to the absolute value of the eigenvector component after a significant change in design variable (25%), which is shown in Table 4. Thus, we see that errors in the eigenvector derivatives computed using the modal expansion are small for eigenvector components which are sensitive to the design variable. When compared to the largest value of an eigenvector component over all of the modes, the errors in the eigenvector derivatives are extremely small. Use of the static correction would improve results further.

Several studies were undertaken to investigate the effect of tolerance on what one considers a repeated eigenvalue, or a repeated eigenvalue derivative. It was universally found that a loose definition of repeated, up to a 5% difference, gave a closer comparison to the finite difference computation. This is consistent with the findings for a more complex structure in Ref. 20. The superior numerical performance obtained by considering closely spaced values repeated is due to the tendency of the admixture coefficients produced by Eq. (18) for case 1, and Eq. (31) for case 3 to produce large numbers when the eigenvalues or eigenvalue derivatives are close, respectively. Some very large arbitrary rotations can be generated using these equations for closely spaced modes or derivatives. Thus, for instance, the best results were obtained when mode 6 was considered to have the same eigenvalue derivative as modes 4 and 5, thus including it in case 4.

Table 1 Tube material and cross-sectional properties

5.08-cm-square tube properties	
t_0, mm	0.7747
Area, cm^2	1.550
$I_x = I_y$, cm^4	6.468
J, cm^4	9.698
T-300 Gr/Ep [±60/0]$_s$	
E_{11}, GPa	53.8
G, GPa	20.5
ν	0.32
ρ(kg/m^3)	1799

Table 2 Eigenvalues and derivatives

Mode	f, Hz	$\lambda = \omega^2$	λ', D_1	λ', D_2
1	19.72	15,348.2	1940	3383
2	19.72	15,348.2	2313	3383
3	26.78	28,322.7	4021	5501
4	31.30	38,679.9	0	10,940.8
5	31.30	38,681.5	0	10,942.1
6	31.30	38,681.5	0.7	10,942.1
7	31.30	38,683.6	11,011	10,943.7
8	31.30	38,683.6	16,345	10,943.7
9	61.00	146,921	−732	21,032
10	61.00	146,921	39,981	21,032
11	137.0	741,065	76,712	20,936
12	325.5	4.1833e6	567,305	−808,532
13	373.2	5.4977e6	487,892	545,932
14	373.2	5.4977e6	937,916	545,932
15	500.6	9.8931e6	−52,368	5.1746e6
16	500.8	9.8994e6	183	5.1746e6
17	500.8	9.8994e6	1859	5.1751e6
18	511.7	1.0337e7	1.044e7	5.5768e6

Eigenvector Extrapolation

Many optimization systems use sensitivity information to extrapolate response predictions over a fairly wide range of design variable changes to reduce the number of full finite element analyses performed. This method has been used with very good results in Refs. 1, 3, and 20, for example. Thus, the accuracy of a first-order Taylor series extrapolation is a way to gauge the effectiveness of eigenvector derivative computations for use in optimization schemes. Table 4 compares eigenvector components computed by the finite element code (without reorientation) for $\bar{D}_1 = 1.25$, with components extrapolated from sensitivity information at $\bar{D}_1 = 1$. Also shown are the original eigenvector components (reoriented) computed at $\bar{D}_1 = 1$. Comparisons between the exact and extrapolated eigenvectors are quite good for the larger magnitude components, i.e., for modes 1,2, 7, and 8. The smaller magnitude components give somewhat poorer predictions, but one must keep in mind that modes 4–6 have motions that are largely orthogonal to the optical performance metrics. The kinetic energy in modes 4–6 is largely in local motion of the midspan masses, with only minor motion of the apex mass. Again, comparing the difference in extrapolated and computed values to the largest value of a component of motion over all of the modes, one can see that differences are very small.

Table 3 Eigenvector derivatives with respect to $\bar{D}_1$

Description	Mode	δ_x	δ_y	LOS_x	LOS_y
		Eigenvector derivative components			
Modal expansion					
Cases 1–3	ϕ_1'	0	−0.0725	0	−1.46
Cases 1–3	ϕ_2'	−0.324	0	−1.09	0
Cases 1–4	ϕ_4'	0	−1.2*e*-4	0	9.0*e*-4
Cases 1–4	ϕ_5'	−6.4*e*-7	0	−5.8*e*-4	0
Cases 1–4	ϕ_6'	−0.0120	0	0.0281	0
Cases 1–3	ϕ_7'	0	−1.33	0	−6.80
Cases 1–3	ϕ_8'	1.64	0	8.39	0
Finite difference, $\Delta D_1 = 2\%$					
	ϕ_1'	0	−0.0802	0	−1.58
	ϕ_2'	−0.332	0	−1.06	0
	ϕ_4'	0	−1.7*e*-4	0	2.5*e*-4
	ϕ_5'	−4.0*e*-5	0	−5.0*e*-5	0
	ϕ_6'	−0.0120	0	0.0352	0
	ϕ_7'	0	−1.32	0	−7.32
	ϕ_8'	1.62	0	9.04	0

Table 4 Original eigenvectors vs extrapolated and actually computed eigenvectors

Mode	δ_x	δ_y	LOS_x	LOS_y
	Eigenvector derivative components			
Original eigenvector at $\bar{D}_1 = 1.0$ (reoriented)				
ϕ_1	0	3.542	0	22.30
ϕ_2	3.542	0	22.30	0
ϕ_4	0	−0.01950	0	0.05636
ϕ_5	0.01853	0	−0.05354	0
ϕ_6	0.01300	0	−0.03757	0
ϕ_7	0	−0.01962	0	0.0567
ϕ_8	−0.01591	0	0.04596	0
Extrapolated eigenvector at $\bar{D}_1 = 1.25$				
ϕ_1	0	3.524	0	21.93
ϕ_2	3.459	0	22.03	0
ϕ_4	0	−0.01953	0	0.05658
ϕ_5	0.01853	0	−0.05369	0
ϕ_6	0.01000	0	−0.03054	0
ϕ_7	0	−0.3520	0	−1.642
ϕ_8	0.3941	0	2.144	0
Computed eigenvctor at $\bar{D}_1 = 1.25$ (nonreoriented)				
ϕ_1	0	3.515	0	21.87
ϕ_2	3.459	0	21.98	0
ϕ_4	0	−0.01967	0	0.05579
ϕ_5	0.01705	0	−0.04846	0
ϕ_6	0.01277	0	−0.03772	0
ϕ_7	0	−0.3314	0	−1.609
ϕ_8	0.3512	0	2.103	0

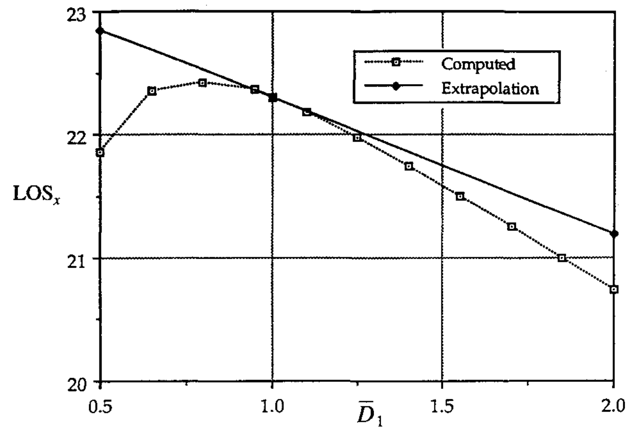

Fig. 3 LOS_x in mode 2 as a function of $\bar{D}_1$.

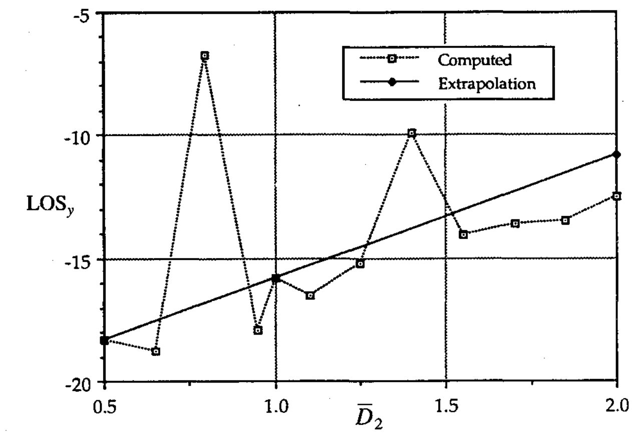

Fig. 4 LOS_y in mode 1 as a function of $\bar{D}_2$, without reorientation.

Mode Reversal

During initial eigenvector calculations it was found that extrapolated eigenvector 1 using sensitivity information with respect to D_1 matched the actually computed eigenvector 1 well when $\bar{D}_1$ was greater than 1, but that the extrapolation matched actual eigenvector 2 for $\bar{D}_1$ less than 1. It was realized that this was due to the eigenvalues crossing in design space at $\bar{D}_1 = 1$ due to their different eigenvalue derivatives. Since the subeigenvalue problem [Eq. (14)] placed the associated eigenvectors in order of ascending eigenvalue derivative, the lowest eigenvalue would remain the lowest for extrapolation in the positive direction but would become the higher of the two for negative extrapolation. A corrective action is to reverse the ordering based on eigenvalue derivative for extrapolations in the negative direction. This situation would become more complicated in multiparameter optimization problems.

Eigenvector Estimates as a Function of $\bar{D}$

The ability of the eigenvector sensitivity procedure to predict mode shape derivatives of closely spaced modes was verified by comparing the first-order Taylor series estimate with computed mode shapes for a range of design variables. The line-of-sight components in x and y were computed for repeated modes 1 and 2, as shown in Figs. 2–5. Figure 2 shows the actual and projected value of LOS_y in mode 1 due to variations in $\bar{D}_1$. The interaction terms between modes 1 and 2 were computed using the formula for case 3, repeated eigenvalues with nonrepeated eigenvalue derivatives. Tracking is seen to be excellent. The estimated eigenvectors have been corrected for the mode reversal behavior below

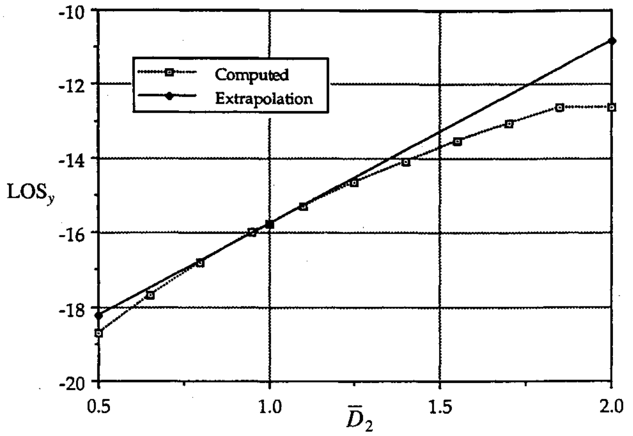

Fig. 5 LOS_y in mode 1 as a function of $\bar{D}_2$, with consistent reorientation.

$\bar{D}_1 = 1$. Figure 3 shows the LOS_x component of mode 2 with respect to changes in D_1, again correcting for mode reversal. Tracking is again excellent. Since modes 1 and 2 have unique eigenvalue derivatives with respect to D_1, the frequencies separate, and there is no ambiguity in the mode shapes requiring reorientation at values of $\bar{D}_1$ above and below 1.

However, this benign behavior is not to be found with respect to design variable 2. Because of symmetry, modes 1 and 2 remain repeated as D_2 changes (case 4), and the reorientation performed on the computed eigenvectors at various values of $\bar{D}_2$ is arbitrary. This is evident in Fig. 4, which shows the component LOS_y in mode 1 with an arbitrary reorientation computation. Tracking of the actual and projected eigenvector component is good only in the mean due to random orientation between modes 1 and 2. In Fig. 5 a consistent reorientation has been applied to computed eigenvectors for all values of $\bar{D}_2$. This eliminates arbitrary rotations, resulting in good mode shape tracking.

V. Conclusions

The modal expansion method allows a simple, numerically stable expression for eigenvector derivatives when eigenvalues are unique, repeated with unique derivatives, or repeated with repeated derivatives. The procedure is computationally inexpensive compared to extensions to Nelson's direct method because no additional inversion of solution set matrices is required. For instance, eigenderivative computation costs for the large truss structure described in Ref. 1 were reduced by a factor of 20 by changing from the direct method to the modal expansion method. The accuracy penalty involved in the modal approximation was only around 5% on average.

In the optimization of structural dynamic response to diverse forcing functions it is usually necessary to include a large number of modes to capture the available energy. This favors the modal expansion method, which allows a large number of modal sensitivities to be computed cheaply. The use of a fairly large number of modes also enhances the accuracy of the modal expansion method since one is closer to convergence than if only a few modes were available.

The use of a specific formula for computation of eigenvector derivatives when frequencies are closely spaced results in improved numerical performance. Adoption of a loose definition of repeatedness avoids the computation of large, erroneous arbitrary rotations between closely spaced modes, and only structurally significant changes in mode shapes based on parameter changes are predicted.

It appears that the modal expansion method should find increasing use for practical engineering optimization problems. The greater efficiency of the method and its consistency with modal analysis methods are both reasons for this. The availability of a solution in the presence of repeated roots should enable the method to be used with increased confidence.

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